On the notion of "ground state" for the nonlinear Schrödinger equation on metric graphs

Séminaire de Mathématiques de Valenciennes

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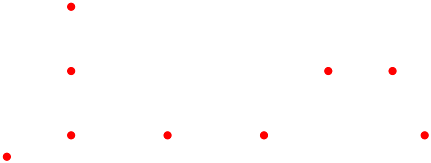
Thursday 15 December 2022

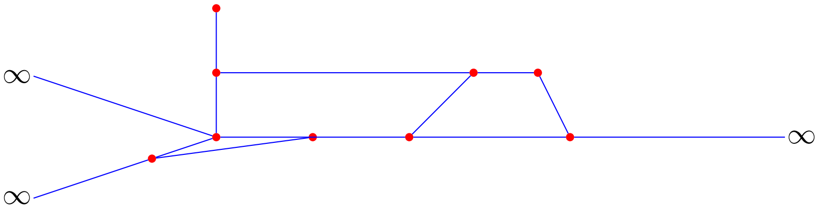
What is a metric graph?

A metric graph is made of vertices



What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



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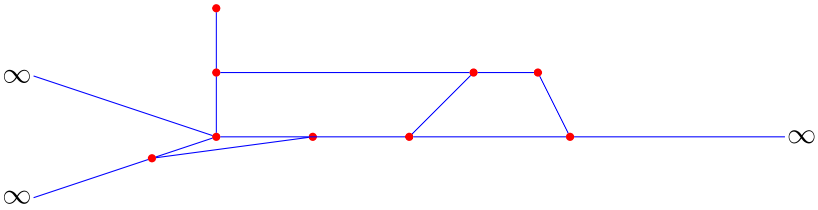

- *metric* graphs: the length of edges are important.

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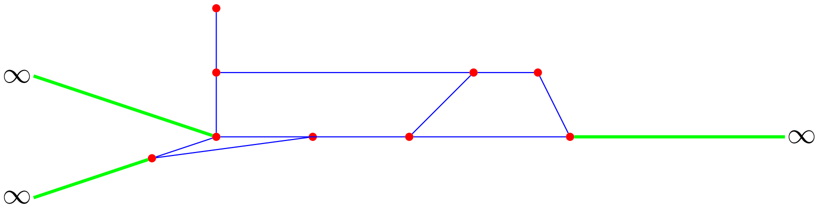
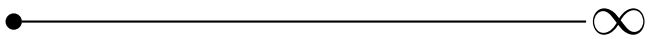


- *metric* graphs: the length of edges are important.
- the edges going to infinity are halflines and have *infinite length*.

Constructions based on halflines



The halfline

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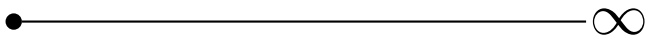

The halfline

The line

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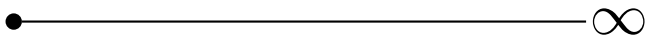

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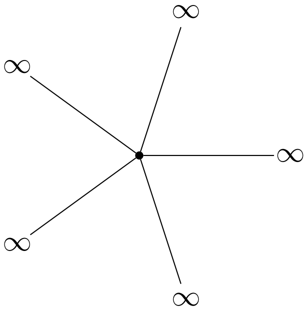

The 5-star graph

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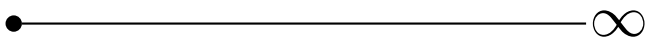

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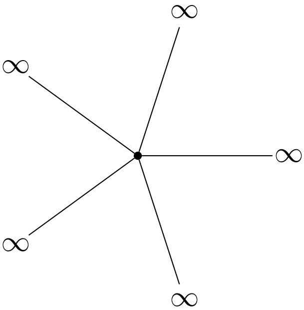

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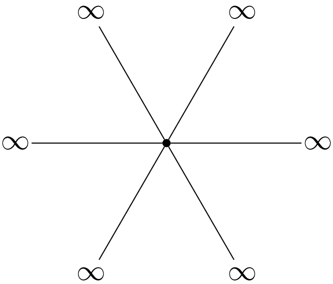
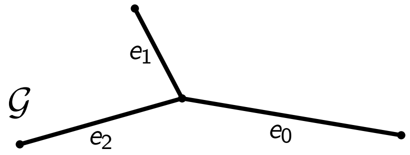


The 6-star graph

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) et e_2 (length 3)

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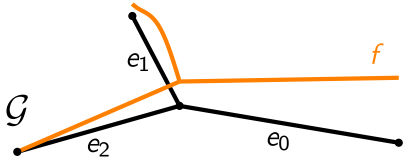

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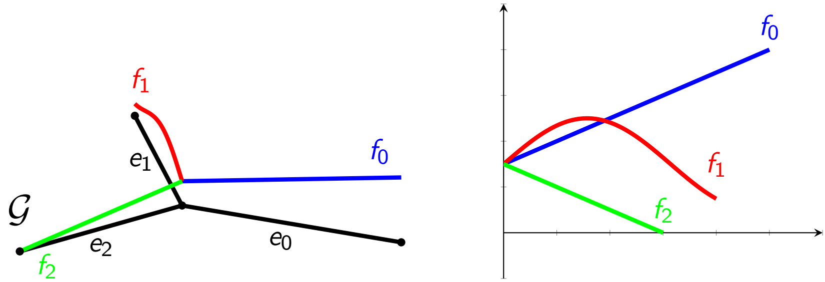

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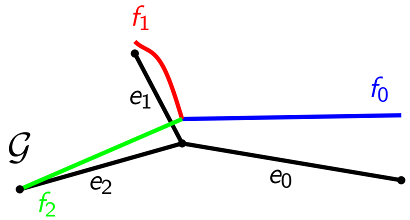

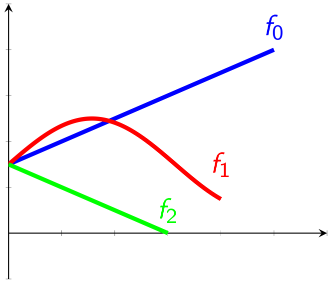

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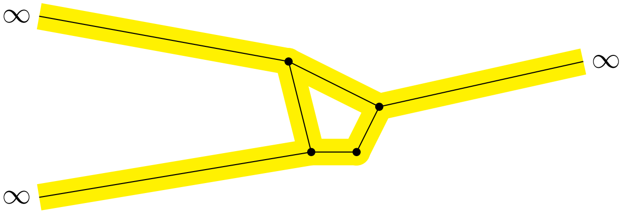
$$\int_{\mathcal{G}} f \, \mathrm{d}x \stackrel{\text{def}}{=} \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs?

Physical motivations

Modeling structures where *only one spatial direction is important.*



A « fat graph » and the underlying metric graph

An application: atomtronics

- A *boson*[1] is a particle with integer spin.

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- This is really remarkable: *macroscopic quantum phenomenon!*
- Since 2000: emergence of *atomtronics*, which studies circuits guiding the propagation of ultracold atoms.

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and we consider the energy minimization problem

$$\inf_{u \in H^1_\mu(\mathcal{G})} \frac{1}{2} \int_{\mathcal{G}} |u'|^2 - \frac{1}{p} \int_{\mathcal{G}} |u|^p,$$

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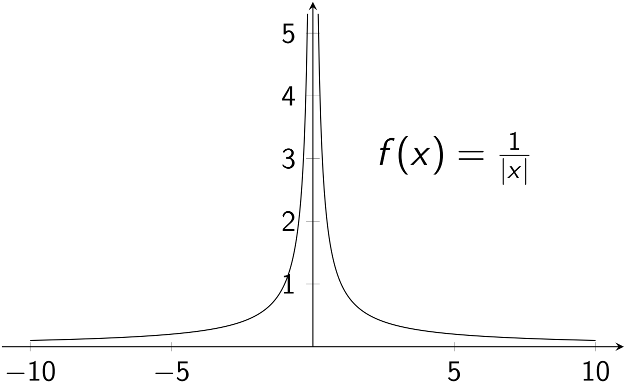
where $2 < p < 6$ (Bose-Einstein: $p = 4$).

Infimum vs minimum



Then

$$\inf_{\mathbb{R}} f = 0$$

but the infimum is not attained (i.e. is not a minimum).

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where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{du}{dx_e}(\mathrm{v})$ is the outgoing derivative of u at v.

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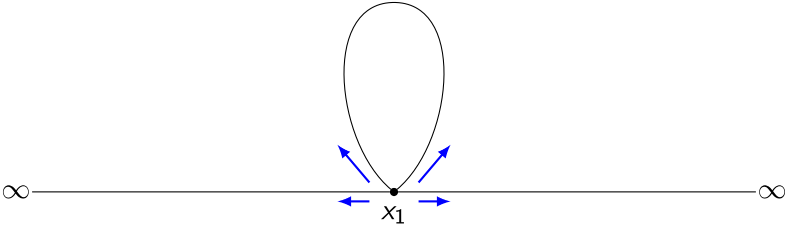
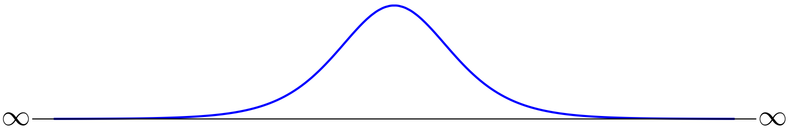
Outgoing derivatives



The real line: $\mathcal{G} = \mathbb{R}$



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where the *soliton* φ_μ is the unique strictly positive, even, and of mass μ solution to an equation of the form

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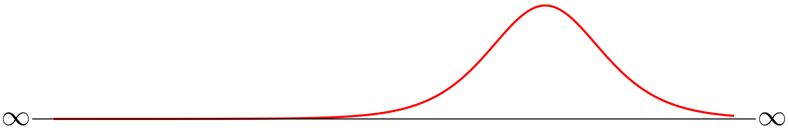

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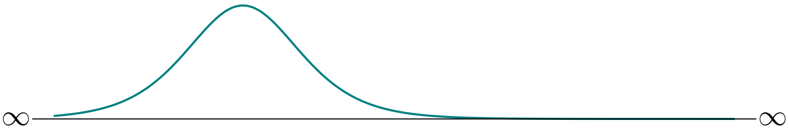

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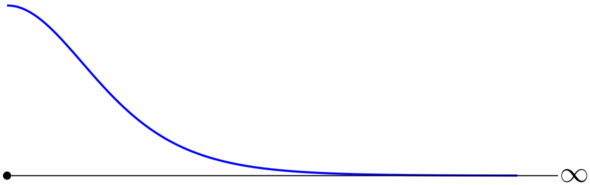
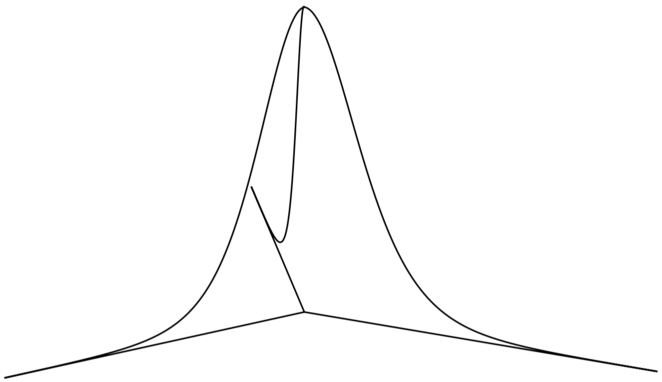
The halfline: $\mathcal{G} = \mathbb{R}^+ = [0, +\infty[$



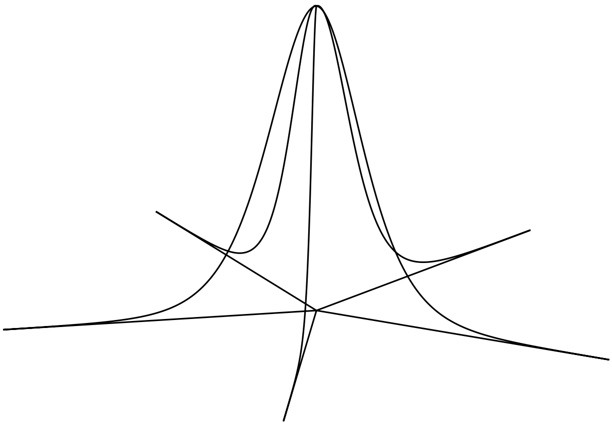
$$\mathcal{S}_\mu(\mathbb{R}^+) = \left\{ \pm\varphi_{2\mu}(x)_{|\mathbb{R}^+} \right\}$$

Solutions are *half-solitons*: no more translations!

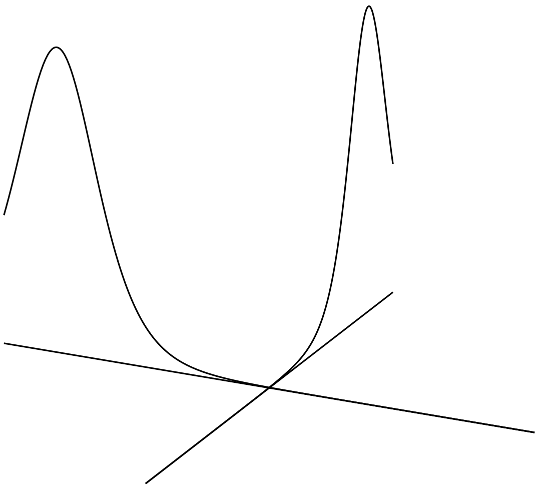
The positive solution on the 3-star graph



The positive solution on the 5-star graph



A continuous family of solutions on the 4-star graph



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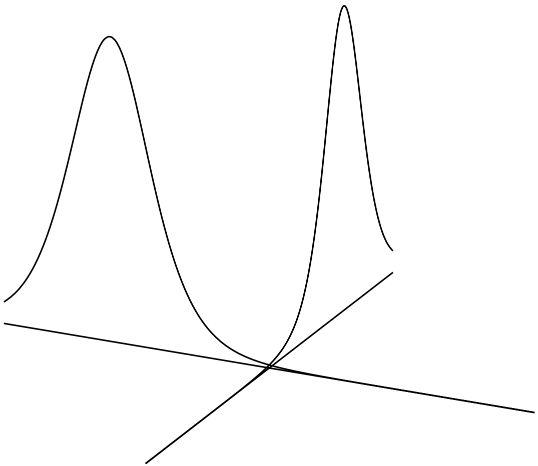

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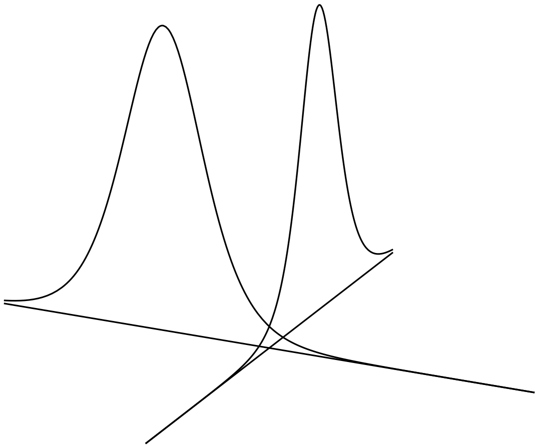

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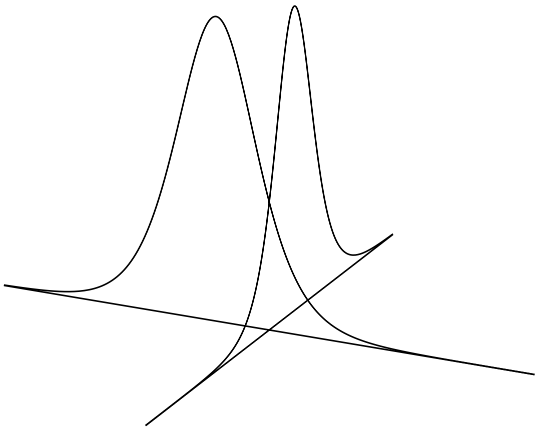

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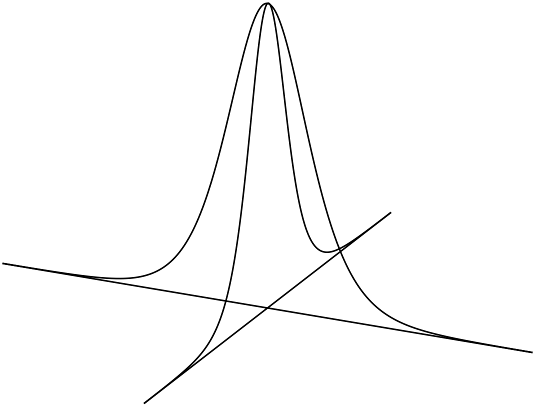

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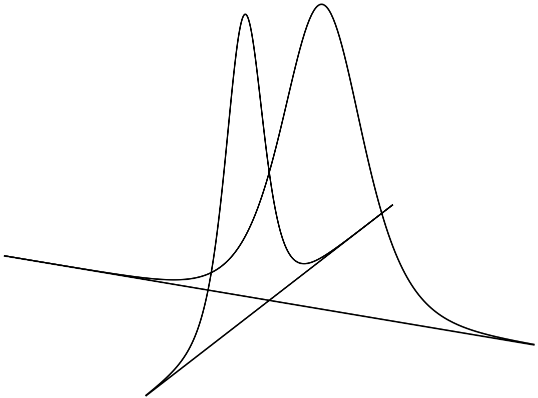

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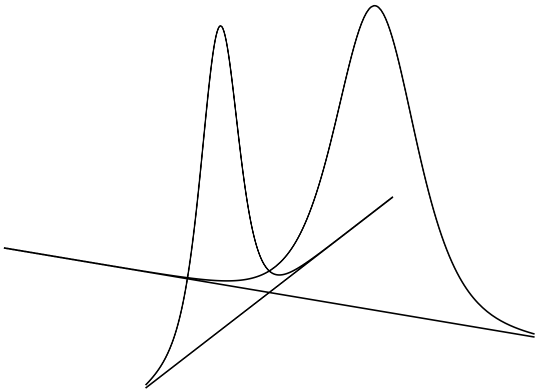

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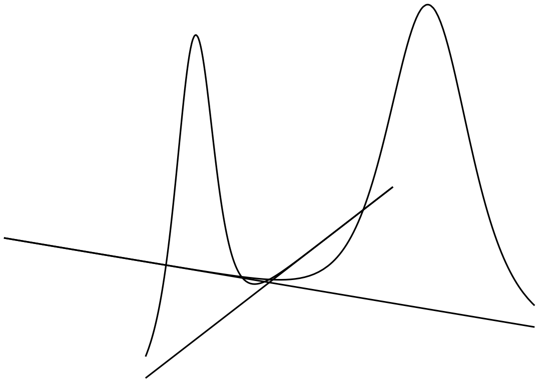

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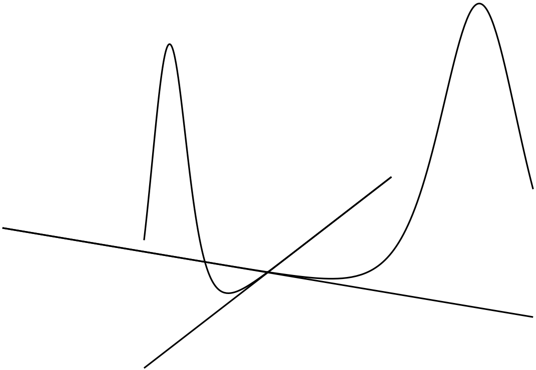

Two energy levels

- The « ground state » energy level is given by

$$c_\mu(\mathcal{G}) = \inf_{u \in H^1_\mu(\mathcal{G})} \frac{1}{2}\int_{\mathcal{G}} |u'|^2 - \frac{1}{p}\int_{\mathcal{G}} |u|^p.$$

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- A *ground state* is a function $u \in H^1_\mu(\mathcal{G})$ with level $c_\mu(\mathcal{G})$. It is a solution of the differential system (NLS).
- We can also consider the minimal level attained by the solutions of (NLS):

$$\sigma_\mu(\mathcal{G}) = \inf_{u \in \mathcal{S}_\mu(\mathcal{G})} \frac{1}{2}\int_{\mathcal{G}} |u'|^2 - \frac{1}{p}\int_{\mathcal{G}} |u|^p.$$

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- A *minimal action solution* of the problem is a solution $u \in \mathcal{S}_\mu(\mathcal{G})$ of the differential problem (NLS) of level $\sigma_\mu(\mathcal{G})$.

Cutting solitons on long edges or halflines

Proposition

Assume that $\mathcal{G}$ has arbitrarily long edges (for instance, if $\mathcal{G}$ has at least one halfline). Then,

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Proof.

□

Four cases

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Four cases

For a N-star graph with $N \geq 3$, we have

$$s_\mu = c_\mu(\mathcal{G}) < \sigma_\mu(\mathcal{G}) = \frac{N}{2} s_\mu.$$

An analysis shows that four cases are possible:

A1) $c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and both infima are attained;

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Question

Are those four cases really possible for metric graphs?

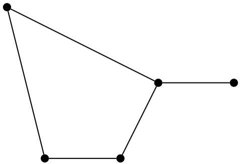
Answer to the question

Theorem (De Coster, Dovetta, G., Serra (to appear))

For every $p \in]2, 6[$, every $\mu > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Case A1

$c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and both infima are attained



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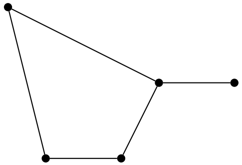

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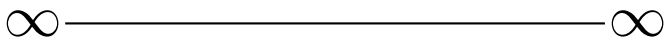

The line

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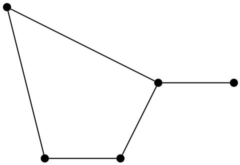

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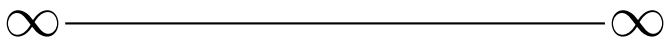

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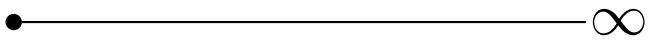

The halfline

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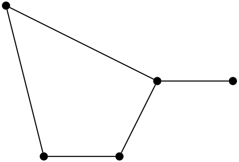

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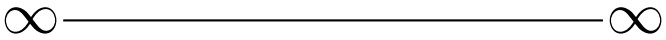

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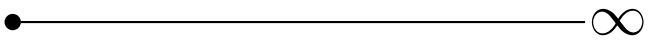

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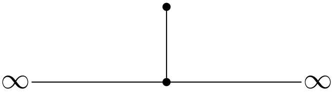
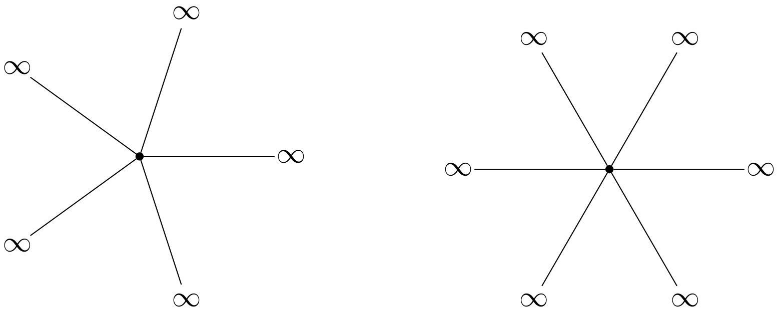


The line with one pendant

Case B1

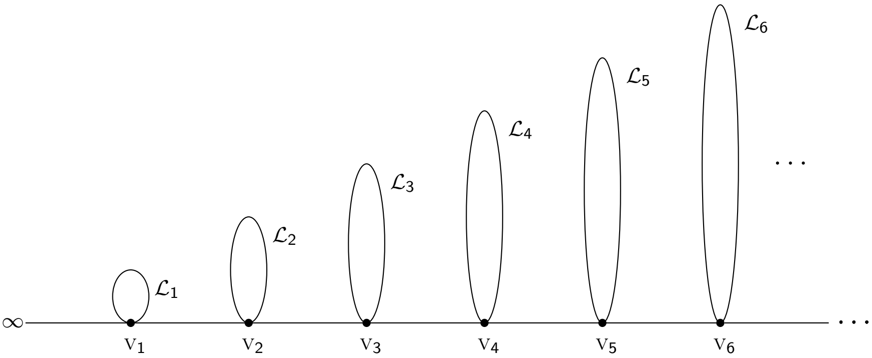
$c_\mu(\mathcal{G}) < \sigma_\mu(\mathcal{G})$, $\sigma_\mu(\mathcal{G})$ is attained but not $c_\mu(\mathcal{G})$



N-star graphs, $N \geq 3$

Case A2

$c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and neither infima is attained



Case B2

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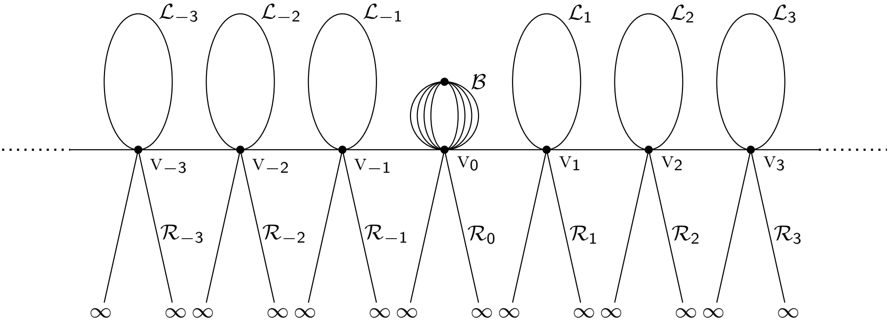

A first existence result

Theorem (Adami, Serra, Tilli 2014)

Let $\mathcal{G}$ be a metric graph with finitely many edges, including at least one halfline. Assume that

$$c_\mu(\mathcal{G}) < s_\mu.$$

Then $c_\mu(\mathcal{G})$ is attained, which means that there exists a ground state, so we are in case A1: $c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$, both attained.

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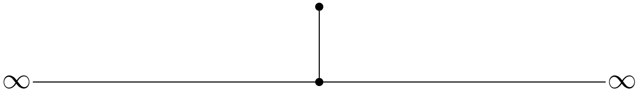
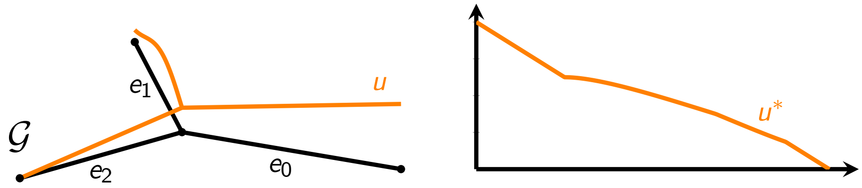
Example:

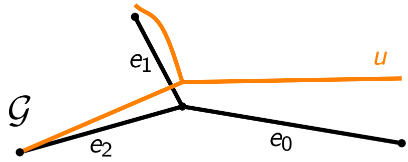


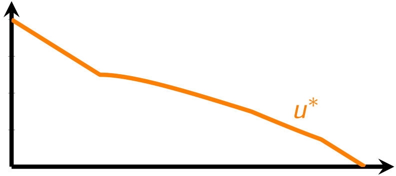
The line with one pendant

Decreasing rearrangement on the halfline



Decreasing rearrangement on the halfline





Fundamental property: for all $t > 0$,

$$\mathrm{meas}_{\mathcal{G}}(\{x \in \mathcal{G}, u(x) > t\}) = \mathrm{meas}_{\mathbb{R}^+}(\{x \in \mathbb{R}^+, u^*(x) > t\}).$$

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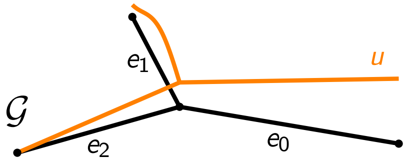

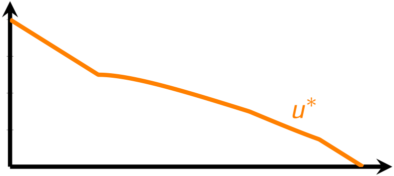

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Consequence: for all $1 \leq p \leq +\infty$,

$$\|u\|_{L^p(\mathcal{G})} = \|u^*\|_{L^p(\mathbb{R}^+)}.$$

The Pólya–Szegő inequality

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Then its decreasing rearrangement u^* belongs to $H^1(0, |\mathcal{G}|)$, and one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

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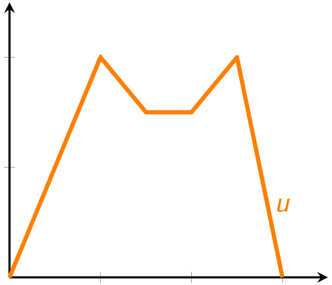
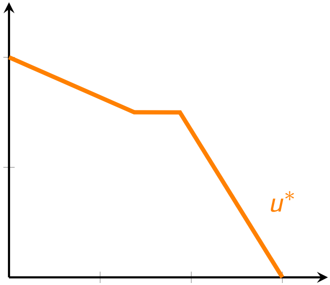
- Friedlander, L. *Extremal properties of eigenvalues for a metric graph*, Ann. Inst. Fourier (Grenoble) **55** (2005) no. 1, 199–211.

The Pólya–Szegő inequality

A simple case: affine functions

We assume that u is piecewise affine.





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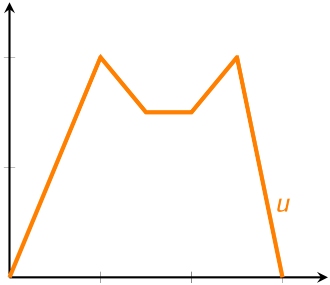

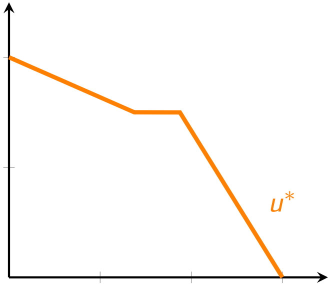

We consider a small open interval $I \subseteq u(\mathcal{G})$ so that $u^{-1}(I)$ consists of a disjoint union of open intervals on which u is affine.

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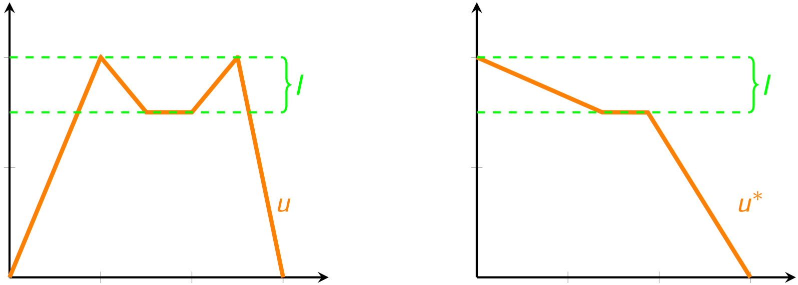

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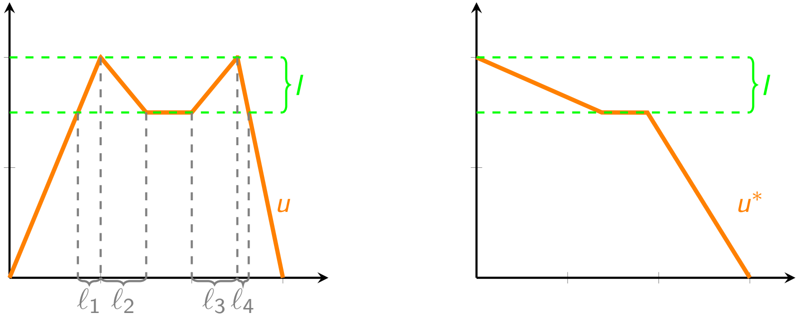

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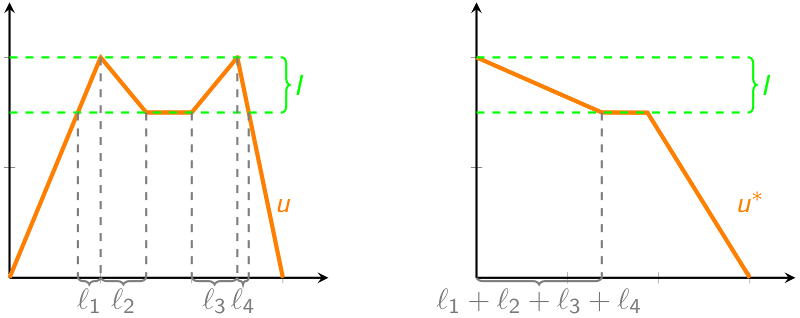

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The Pólya–Szegő inequality

A simple case: affine functions

Original contribution to $\|u'\|_{L^2}^2$:

$$A := \ell_1 \frac{|I|^2}{\ell_1^2} + \ell_2 \frac{|I|^2}{\ell_2^2} + \ell_3 \frac{|I|^2}{\ell_3^2} + \ell_4 \frac{|I|^2}{\ell_4^2}$$

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Contribution to $\|(u^*)'\|_{L^2}^2$:

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The Pólya–Szegő inequality

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Inequality between arithmetic and harmonic means:

$$\frac{\ell_1 + \ell_2 + \ell_3 + \ell_4}{4} \geq \frac{4}{\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} + \frac{1}{\ell_4}}$$

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A refined Pólya–Szegő inequality...

... or the importance of the number of preimages

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Let $N \geq 1$ be an integer. Assume that, for almost every $t \in]0, \|u\|_\infty[$, one has

$$u^{-1}(\{t\}) = \{x \in \mathcal{G} \mid u(x) = t\} \geq N.$$

Then one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \frac{1}{N^2}\|u'\|_{L^2(\mathcal{G})}.$$

Assumption (H)

Definition (Adami, Serra, Tilli 2014)

We say that a metric graph $\mathcal{G}$ satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty[\to \mathcal{G}$ parameterized by arclength, with disjoint images except for an at most countable number of points, and such that $\gamma_1(0) = \gamma_2(0) = x_0$.

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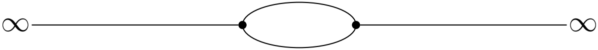

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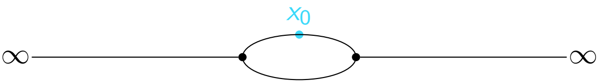

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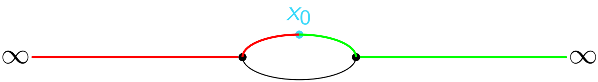

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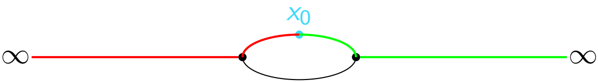

Consequence: *all* nonnegative $H^1(\mathcal{G})$ functions have at least two preimages for almost every $t \in]0, \|u\|_\infty[$.

Non-existence of ground states

Theorem (Adami, Serra, Tilli 2014)

If a metric graph $\mathcal{G}$ has at least one halfline and satisfies assumption (H), then

$$c_\mu(\mathcal{G}) := \inf_{u \in H^1_\mu(\mathcal{G})} E(u) = s_\mu$$

but it is never achieved

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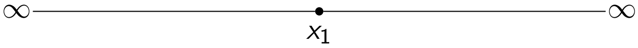
but it is never achieved, unless $\mathcal{G}$ is isometric to one of the exceptional graphs depicted in the next few slides.

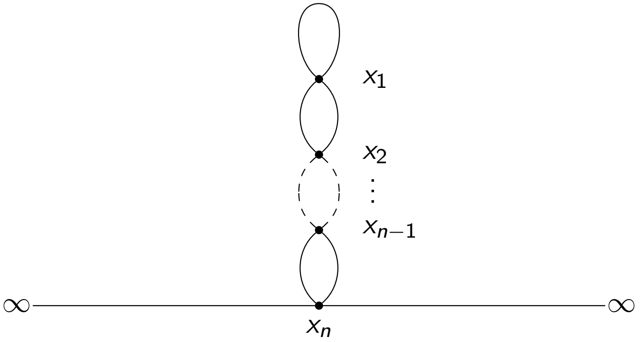
Non-existence of ground states

Exceptional graphs: the real line



Non-existence of ground states

Exceptional graphs: the real line with a tower of circles



A doubly constrained variational problem

Compactness

We define

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}$$

where e is a given bounded edge of $\mathcal{G}$

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$$c_\mu(\mathcal{G}, e) := \inf_{u \in H^1_\mu(\mathcal{G}) \cap X_e} E(u).$$

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Theorem

There exists $\overline{R} > 0$ depending only on μ and p such that, if $\mathcal{G}$ satisfies assumption (H) with a bounded edge e of length $R \geq \overline{R}$, then $c_\mu(\mathcal{G}, e)$ is attained.

A doubly constrained variational problem

An existence result

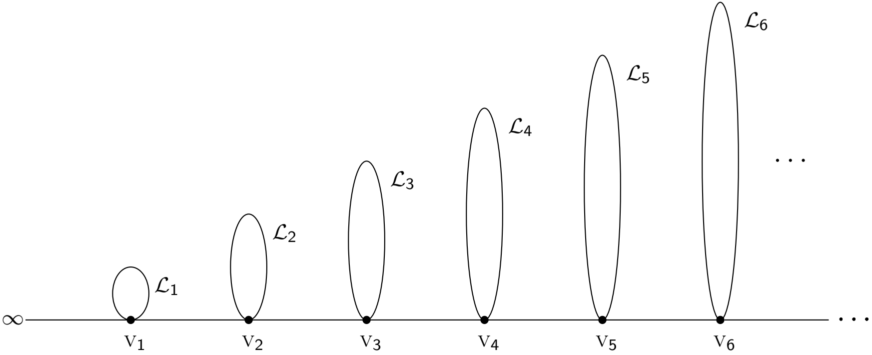
Theorem

Let $\mathcal{G}$ satisfy assumption (H) with a bounded edge e of length R and $\ell_0 \leq \inf_{e \in E} |e|$. There exists $\widetilde{R} \geq \overline{R}$ (with $\overline{R}$ given by the previous Theorem) depending only on ℓ_0, μ and p such that if $R \geq \widetilde{R}$ and u is a minimizer for $c_\mu(\mathcal{G}, e)$, then $u \in \mathcal{S}_\mu(\mathcal{G})$ and $u > 0$ or $u < 0$ on $\mathcal{G}$. Moreover,

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G} \setminus e)}.$$

What's going on in case A2?

$c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G})$ and neither infima is attained



What's going on in case A2?

Using the previous results

- Since $\mathcal{G}$ has at least one halfline and satisfies assumption (H), one has $c_\mu(\mathcal{G}) = s_\mu$ and the infimum is not attained (as $\mathcal{G}$ does not belong to the class of exceptional graphs).

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- One obtains

$$s_\mu = c_\mu(\mathcal{G}) \le \sigma_\mu(\mathcal{G}) \le \liminf_{n\to\infty} c_\mu(\mathcal{G}, \mathcal{L}_n) = s_\mu,$$

so

$$c_\mu(\mathcal{G}) = \sigma_\mu(\mathcal{G}) = s_\mu$$

and neither infimum is attained.

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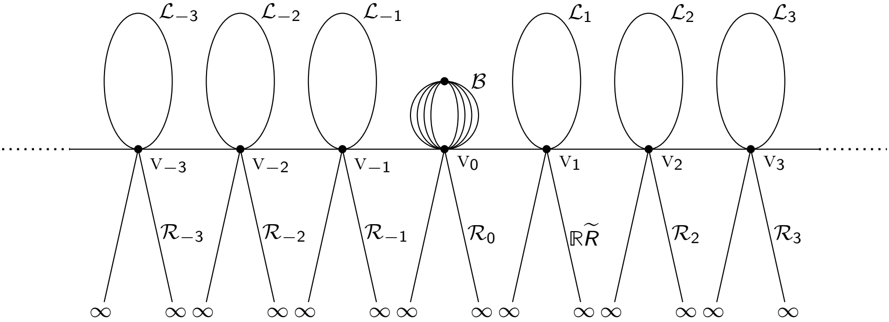
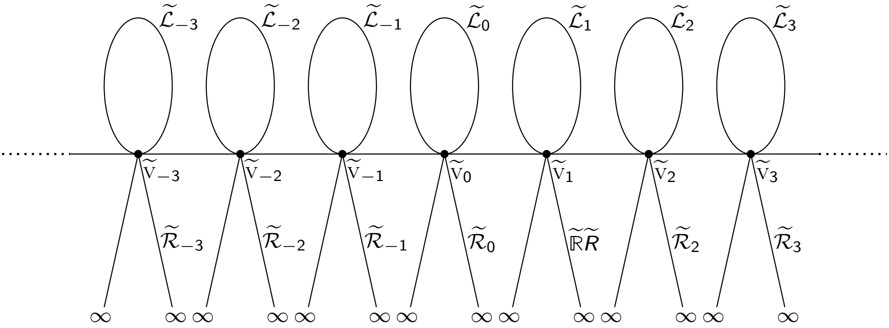


The graph $\mathcal{G}_N$.

The loops $\mathcal{L}_i$ have length N and $\mathcal{B}$ is made of N edges of length 1.

What's going on in case B2?

A second, periodic, graph



The graph $\widetilde{\mathcal{G}}_N$.

The loops $\widetilde{\mathcal{L}}_i$ have length N.

What's going on in case B2?

Two problems at infinity

- Since $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$ satisfy (H) and contain halflines, one has

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- Therefore, for large N, we have that

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Mathematical motivations

Main message

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Replacing $\mathcal{G}$ by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur.

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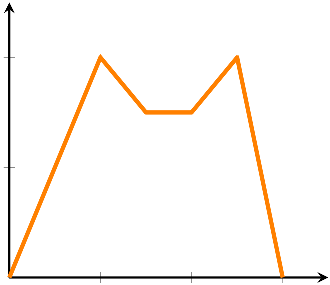
Thanks for your attention!

Merry Christmas!



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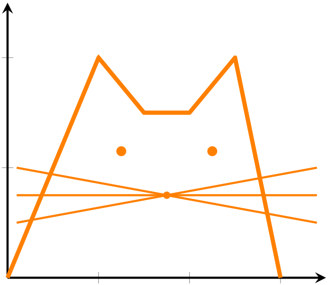

Main papers

Adami, R., Serra, E., Tilli, P. *NLS ground states on graphs* Calculus of Variations and Partial Differential Equations, 54(1), 743-761 (2015).

De Coster C., Dovetta S., Galant D., Serra E. *On the notion of ground state for nonlinear Schrödinger equations on metric graphs* To appear.

Overviews of the subject

- Adami R., Serra E., Tilli P. *Nonlinear dynamics on branched structures and networks* https://arxiv.org/abs/1705.00529 (2017)
- Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs* J. Phys. A: Math. Theor. 55 243001 (2022)
- Adami R. *Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE)* https://www.youtube.com/watch?v=G-FcnRVvoos (2020)

Why $p < 6$?

Given $u \in H^1(\mathbb{R})$, one has a one-parameter family of L^2-norm preserving scalings $u \mapsto u_t$, where $u_t(x) := t^{1/2}u(tx)$. Direct computations show that

$$\|u_t'\|_{L^2}^2 = t^2 \|u'\|_{L^2}^2, \qquad \|u_t\|_{L^p}^p = t^{\frac{p}{2}-1} \|u\|_{L^p}^p.$$

Hence,

$$E(u_t) = \frac{1}{2}\|u_t'\|_{L^2}^2 - \frac{1}{p}\|u_t\|_{L^p}^p = \frac{t^2}{2}\|u_t'\|_{L^2}^2 - \frac{t^{\frac{p}{2}-1}}{p}\|u_t\|_{L^p}^p.$$

If $p > 6$, the term with the negative sign wins, hence *the energy functional is not bounded under the mass constraint.* For more information about the $p \geq 6$ case, see e.g.

Chang X., Jeanjean L., Soave N. *Normalized solutions of L^2-supercritical NLS equations on compact metric graphs* https://arxiv.org/abs/2204.01043 (2022)